



MATHEMATICS SPECIALIST : UNITS 1 & 2, Tuesday 19th September 2023

Test 4 – (10%)

2.2.1-2.2.11, 1.1.2, 2.3.2-2.3.3

KB

Time Allowed 25 minutes	First Name	Surname	Marks
			26 25 marks

Circle your Teacher's Name: Ms Chua Mr Koulianos Mr Liu Ms Robinson

Assessment Conditions: (N.B. Sufficient working out must be shown to gain full marks)

- ❖ Calculators: Allowed
- ❖ Formula Sheet: Provided
- ❖ Notes: Not Allowed

PART B – CALCULATOR ASSUMED

1. [3, 2 and 2 marks]

O(0,0), A(6,1) and B(5,3) are the vertices of a triangle.

- a. Triangle OAB is transformed to triangle OA'B', where A' is (6, 19) and B' is (5, 18), by transformation T₁. Find and describe matrix T₁.

$$T_1 \begin{pmatrix} 6 & 5 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 19 & 18 \end{pmatrix} \quad \checkmark$$

1 mark Write Matrix Equation
1 mark Correct T

$$T_1 = \begin{pmatrix} 6 & 5 \\ 19 & 18 \end{pmatrix} \begin{pmatrix} 6 & 5 \\ 1 & 3 \end{pmatrix}^{-1}$$

1 mark Description

$$= \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \quad \checkmark \text{ shear parallel to y axis } (SF3) \text{ comment only}$$

- b. Triangle OA'B' is transformed by matrix T₂ = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ to triangle OA''B''. Determine points A'' and B''.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 6 & 5 \\ 19 & 18 \end{pmatrix} = \begin{pmatrix} 19 & 18 \\ 6 & 5 \end{pmatrix} \quad \checkmark$$

1 mark Mult. by T₂

$$A'' = (19, 6) \quad B'' = (18, 5) \quad \checkmark$$

1 mark Gives co-ord

- c. Find a single matrix that will transform triangle OAB directly to triangle OA''B''.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} \quad \checkmark$$

$$\begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -3 \end{pmatrix} \quad \checkmark$$

1 mark Correct order

1 mark Answer inverse

2. [1, 1, 1, 2 and 2, marks]

A brewer uses two different malts, M1 and M2, and a special ingredient, factor B, to produce two beers, Swan Bay and Crawley Bars. The number of units of each ingredient required to produce these beers are given in the table.

	M1	M2	B
1 litre of Swan Bay	4	3	1
1 litre of Crawley Bars	5	2	1

a. Write a matrix, A, which represents the information in the table.

$$\begin{pmatrix} 4 & 3 & 1 \\ 5 & 2 & 1 \end{pmatrix}$$

Orders for Swan Bay and Crawley Bars are 20 000 litres and 30 000 litres respectively for the next production period.

b. Write a row matrix, B, representing the size of orders.

$$(20\ 000 \quad 30\ 000)$$

The cost of ingredients M1, M2 and B are 10 cents, 15 cents and 50 cents per unit respectively.

c. Write a column matrix, C, representing the cost of the three ingredients.

$$\begin{pmatrix} 0.1 \\ 0.15 \\ 0.5 \end{pmatrix}$$

Determine, using matrices A, B and C and appropriate matrix operations:
 * For d, e, f 1 mark for answer only.

d. the matrix which represents the total requirements of M1, M2 and B for the upcoming order.

$$(20\ 000 \quad 30\ 000) \begin{pmatrix} 4 & 3 & 1 \\ 5 & 2 & 1 \end{pmatrix} = (230\ 000 \quad 120\ 000 \quad 50\ 000)$$

1 mark Shows matrix operation

1 mark Answer

e. the matrix which represents the cost of 1 litre of each type of beer.

$$\begin{bmatrix} 4 & 3 & 1 \\ 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.15 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1.35 \\ 1.30 \end{bmatrix}$$

1 mark Shows matrix operation

1 mark Answer

- f. the matrix which represents the total cost of the upcoming order.

$$\begin{bmatrix} 20 & 000 & 300 & 000 \end{bmatrix} \begin{bmatrix} 1.35 \\ 1.30 \end{bmatrix} = \begin{bmatrix} 66 & 000 \end{bmatrix}$$

1 mark Matrix Operation

1 mark Answer

3. [4 marks]

Prove by contradiction that there are no positive integer solutions to the Diophantine

$$\text{equation } x^2 - y^2 = 1.$$

Assume that $\exists x, y \in \mathbb{Z}^+$ such that

$$x^2 - y^2 = 1. \quad \checkmark$$

$$x^2 - y^2 = 1$$

$(x+y)(x-y) = 1$ Only factors of 1 are ± 1 $\checkmark$

$$\therefore x+y = 1 \quad \text{or}$$

$$x-y = 1$$

$x+y = -1$ 1 mark Sets up supposition

$x-y = -1$ 1 mark Factors and identifies factors of 1

Solve simultaneously

$$2x = -2$$

$$x = -1$$

$$y = 0$$

Neither $-1, 0$ are positive integers

$$\therefore 1+y = 1$$

$$y = 0.$$

0 is not a positive integer $\checkmark$

$\therefore$ Thus x and y cannot be elements of $\mathbb{Z}^+$

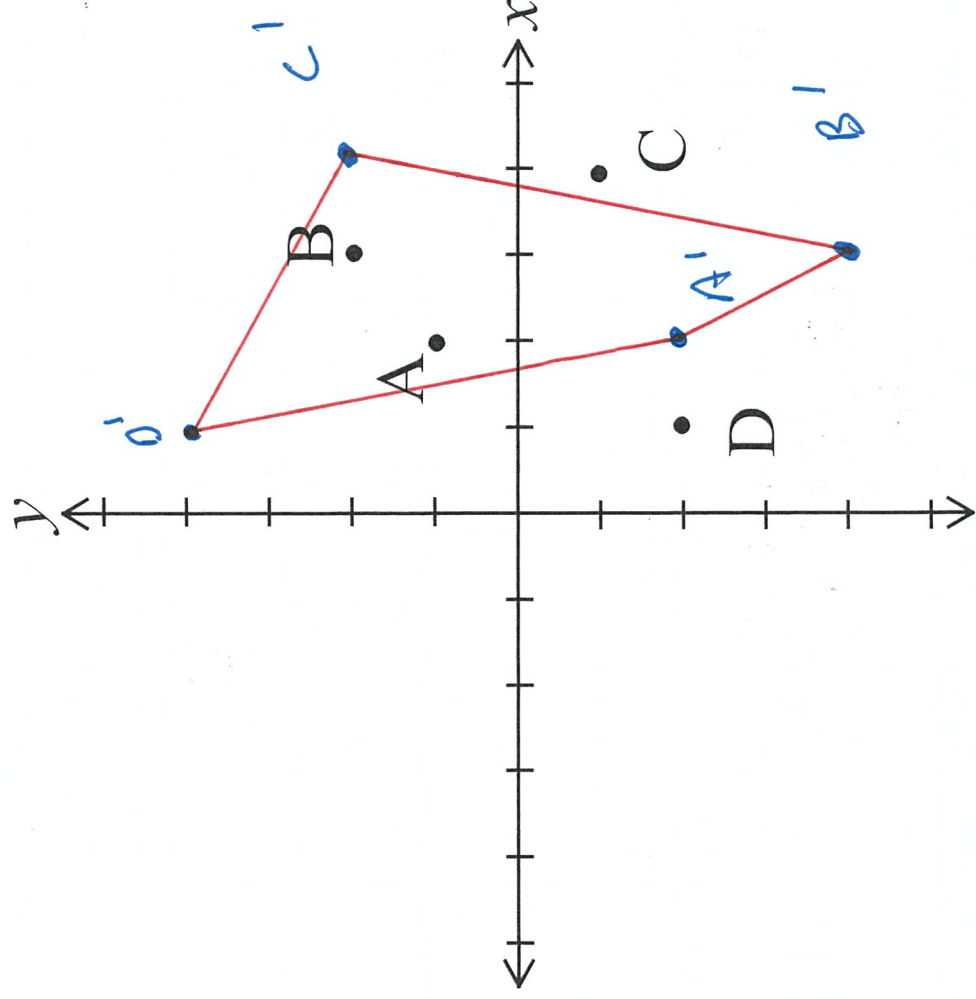
and contradicts our assumption

$\therefore$ no positive integer solution to $x^2 - y^2 = 1$ $\checkmark$

4. [2, 2, 1 and ~~2~~ marks]

A quadrilateral ABCD has vertices:

A (2, 1) B (3, 2) C (4, -1) D (1, -2)



Grid!

d. Sketch on the axes given, the image of ABCD after the transformation S has been

applied, where $S = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 & 1 \\ -2 & -4 & -2 & -4 \end{bmatrix}$$

1 mark Points (can be in matrix)

1 mark Plotted (do not penalise if unconnected)

e. Use words to describe the transformation.

Dilation along y axis $SF = 2$

Reflection in x axis

(SF of -2 no marks)

f. If the area of ABCD is k units, state the area of its image under S.

$$|\text{Det } S| = 2 \quad \therefore 2k$$

g. State the matrix that would transform the image back to ABCD.

$$\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

Reading Time: An initial 2 minutes to view BOTH sections



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PART A – CALCULATOR FREE

1. [1, 2 and 1 marks]

Consider the matrices below:

$$A = \begin{bmatrix} 0 & 3 & 1 \\ 4 & -1 & 5 \\ 1 & -2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -5 \\ -9 & 3 \\ 11 & 3 \end{bmatrix}$$

a. Why is BA not possible?

Columns of B $\neq$ Rows of A ✓ r/w

b. If $A^{-1} = \frac{-1}{16} \begin{bmatrix} 8 & -8 & 16 \\ -3 & -1 & y \\ x & 3 & -12 \end{bmatrix}$, give the values of x and y.

$$A A^{-1} = I$$

$$\therefore 0 \times 8 + 3 \times -3 + 1 \times x = -16 \times 1$$

$$-9 + x = -16$$

$$x = -7 \quad \checkmark \quad \text{r/w}$$

$$3y - 12 = -16 \times 0$$

$$y = 4 \quad \checkmark \quad \text{r/w}$$

c. A third matrix C is such that $BC = \begin{bmatrix} 7 \\ 21 \\ -39 \end{bmatrix}$. Give the dimensions of C.

$$B_{3 \times 2} \times C_{n \times m} = D_{3 \times 1}$$

$$\therefore C \text{ is } 2 \times 1 \quad \checkmark \quad \text{r/w}$$

2. [2 marks]

Express $10.\overline{07}$ as an improper fraction.

$$100x = 1007.0707\ldots \quad \checkmark$$

$$x = 10.0707\ldots \quad \checkmark$$

$$99x = 997$$

$$x = \frac{997}{99} \quad \checkmark$$

1 mark Two equivalent equations

1 mark Answer

3. [3 and 3 marks]

Janice claims that if two matrices A and B are such that $AB = I$, where I is the Identity matrix

then A and B must be inverses of each other.

a. Is Janice's claim, correct? If so, explain why, if not explain why not.

Yes $\checkmark$

$$AB = I$$

$$AB = I$$

1 mark Yes

$$A^{-1}AB = A^{-1}I$$

$$ABB^{-1} = IB^{-1}$$

1 mark Shows $B = A^{-1}$

$$IB = A^{-1}$$

$$AI = B^{-1}$$

1 mark shows $A^{-1} = B$

$$B = A^{-1} \quad \checkmark$$

$$A = B^{-1} \quad \checkmark$$

Note: must pre or post multiply correctly to get final result. Each statement / proof must start with $AB = I$

b. Given that two matrices A and B are inverses of each other, simplify the expression below, showing each step.

$$A^{-1}B^2A^3B^{-1}$$

$$B B^2 A^3 A \quad \checkmark$$

1 mark Converts A^{-1} and B^{-1}

$$= BB \underline{BA} AA \quad \checkmark$$

1 mark Shows $BA = I$

$$= BB I AA \quad \checkmark$$

1 mark Final answer, accept B^{-1}

$$= A \quad \checkmark$$

4. [2 and 3 marks]

a. Given Matrix $A = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix}$, show that $A^2 = I$.

$$\begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix} = \begin{bmatrix} \cos^2 \theta + \sin^2 \theta & \cos \theta \sin \theta - \sin \theta \cos \theta \\ \sin \theta \cos \theta - \cos \theta \sin \theta & \sin^2 \theta + \cos^2 \theta \end{bmatrix} \checkmark$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

1 mark Write matrix A^2
1 mark simplifies to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b. Given Matrix $B = \begin{bmatrix} x & 1+x \\ 1-x & -x \end{bmatrix}$, state any restriction on x , if any, for $B^2 = I$.

$$\det A = -x^2 - (1-x^2) = -1$$

1 mark Finds $\det A$

Since $\det A \neq 0 \forall x$ 1 mark States restriction
no restriction on x 1 mark with reason

$$B^2 = \begin{bmatrix} x^2 + (1+x)(1-x) & x(1+x) - x(1+x) \\ x(1+x) - x(1-x) & (1-x)(1-x) + x^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \therefore$$

5. [4 marks]

Solve the equation to determine matrix B :

$$\begin{bmatrix} 3 & 0 \\ 5 & 0 \end{bmatrix} B = B + \begin{bmatrix} 2 & 4 \\ 6 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 5 & 0 \end{bmatrix} B - B = \begin{bmatrix} 2 & 4 \\ 6 & 7 \end{bmatrix}$$

$$\left[\begin{bmatrix} 3 & 0 \\ 5 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] B = \begin{bmatrix} 2 & 4 \\ 6 & 7 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 2 & 0 \\ 5 & -1 \end{bmatrix} B = \begin{bmatrix} 2 & 4 \\ 6 & 7 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 0 \\ 5 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 4 \\ 6 & 7 \end{bmatrix} \checkmark$$

$$= -\frac{1}{2} \begin{bmatrix} -1 & 0 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \checkmark$$

1 mark Finds B

6. [3 and 2 marks]

- a. Find a single 2×2 matrix which describes a rotation Of 135° anticlockwise followed by another rotation of 30° anticlockwise.

$$\begin{pmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{pmatrix} \begin{pmatrix} \cos 135 & -\sin 135 \\ \sin 135 & \cos 135 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

1 mark uses correct rotational matrices order not important
1 mark substitutes exact values
1 mark Final matrix

$$= \begin{pmatrix} \frac{-\sqrt{3}-1}{2\sqrt{2}} & \frac{-\sqrt{3}+1}{2\sqrt{2}} \\ \frac{-1+\sqrt{3}}{2\sqrt{2}} & \frac{-\sqrt{3}-1}{2\sqrt{2}} \end{pmatrix}$$

- b. Hence find the values of $\sin\left(\frac{11\pi}{12}\right)$ and $\cos\left(\frac{11\pi}{12}\right)$

$$\sin\left(\frac{11\pi}{12}\right) = \sin(165^\circ) = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\cos\left(\frac{11\pi}{12}\right) = \cos(165^\circ) = \frac{-\sqrt{3}-1}{2\sqrt{2}}$$

1 mark Finds value $\sin\left(\frac{11\pi}{12}\right)$

1 mark Finds value $\cos\left(\frac{11\pi}{12}\right)$

Note if students express combined transformation in terms of $\cos/\sin(135, 30)$ 2/3 for part (a) and 1/2 part (b), If they answer part (a) with a single rotational matrix in terms of $\cos/\sin(165)$ then 1/3.